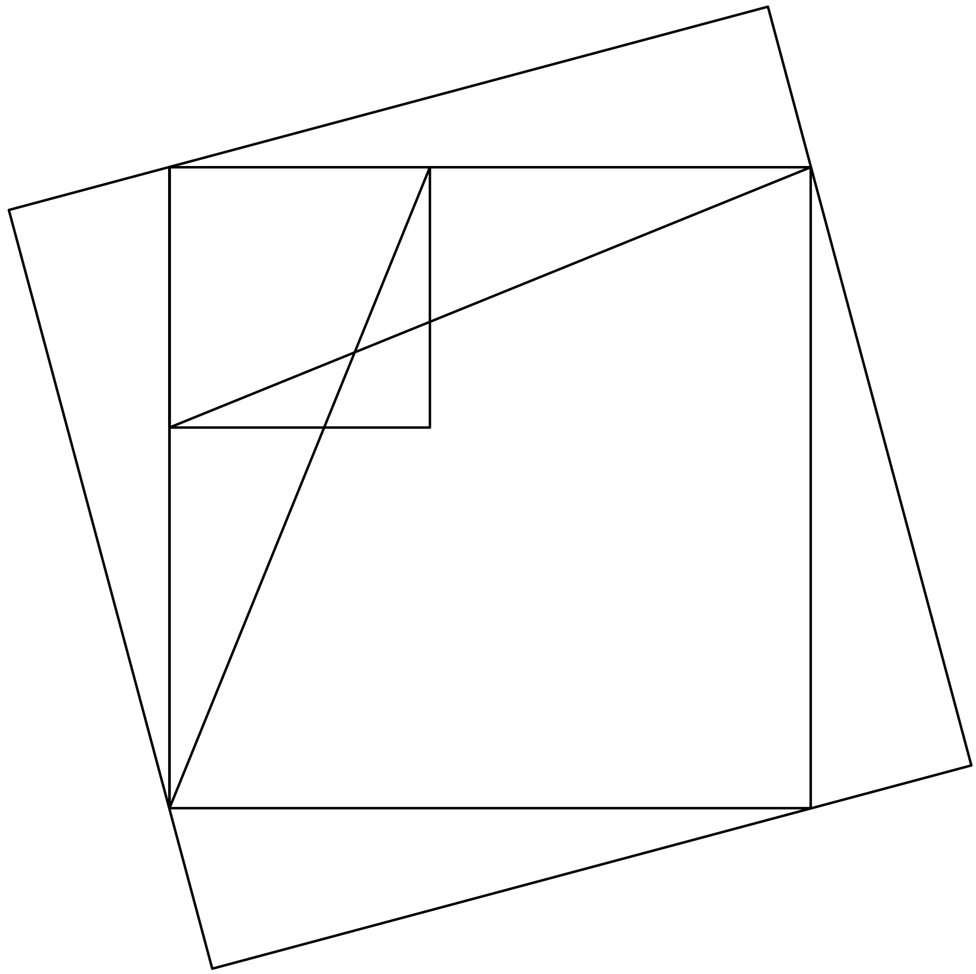

Patterns of Mathematical Argument

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Contents

Using This Text

This is a short book about cultivating virtues. It aims at encouraging curiosity, clarity, and community. Those three, in concert, allow for and encourage mathematical excellence. The way to encourage those virtues in your students is to confuse them: to give them hard problems –problems they don't know how to solve– that will tax their intellectual and creative muscles.

So this text is a selection of good, hard problems I've used with my own students. For each problem, I've included some annotations to help you use them in the classroom:

1. A note on the kairos. The *kairos* is the critical moment. Give a problem too early and it frustrates students to no good result. Give it too late and it's trivial. I explain where I fit each of these into my curriculum.
2. An explanation of how much class and homework time you can expect the problem to take. These numbers aren't absolute; different classes with different personalities will react subtly differently to each problem, but this approximation can help guide your planning.
3. The answer to each question (well, except one, which I suspect is an open problem). If you want to work these on your own, just look at the student page before flipping to the teacher notes!
4. Discussion notes. I recount things my students have done and said in discussion that it might be nice to think about ahead of time. Certainly they will surprise you in a host of ways.
5. Sample student writing. Much of this is cleaned up from the raw writing that I receive, so take these as model papers. Actual student work is messier and usually longer. I also never require students to type their papers, as TEX is time-consuming for students to learn and other typesetting systems produce worse results than handwriting.

The Classroom Liturgy

I do what this text describes about fifteen times in Pre-Calculus and about ten times in Algebra II. You could talk me into adding a few more, and you can definitely start with fewer. It's not the only thing we do, but it's an important thing. Here's the ideal rhythm of a discussion week in my classroom:

On a Monday I give the students a novel and difficult problem that they don't know how to solve. They wrestle with it for the next forty-five minutes. Tuesday they wrestle with the problem all forty-five minutes, seeing a promising avenue of attack that ultimately fails. Wednesday they come to grips with a decent plan to solve the problem, and run it to ground. Wednesday night they start writing their papers, which are due Monday next. Thursday I introduce some new grammar or notation and we practice it. Friday we practice the notation a bit more, perhaps running into something perplexing that might lead to the next novel and difficult problem.

Wash, rinse repeat.

There are a few parts of that weekly liturgy that would have terrified me as a young math teacher. “You let your students work on a single problem for forty-five minutes?!” Yep. “You spend a whole day on work that goes nowhere?” Regularly. “How on earth do you plan your lessons if you don’t know how long a particular problem will take?” Flexibly. “You assign papers?” And grade them too. “What do *you* do while they’re working these problems?” Mostly hide in the corner and try not to say anything disruptive, which is surprisingly hard. On my best days, I’ll say five or six carefully chosen sentences (maybe four of which are questions) frown once or twice, and maybe grunt appreciatively. On my worst days, I spoil the problem by giving the students too much help and robbing them of the interesting work. It’s tricky teaching. Sometimes I have to leave the room to keep myself from ruining things. My students find that pretty funny.

“All well and good”, you might say, “but how would I actually *do* this in my classroom? I don’t have a classroom of genius-mathematician-scholar-writers.” Neither do I. But you can create a culture that allows students to grow into, if not math geniuses, at least competent mathematical thinkers.

Talking About Mathematics

The central image I use when talking to my students about how to form their mathematical community is the body of Christ. As St. Paul writes in his first letter to the Corinthians, chapter 12:

> [T]here are varieties of gifts, but the same Spirit; and there are varieties of service, but the same Lord; and there are varieties of activities, but it is the same God who empowers them all in everyone. To each is given the manifestation of the Spirit for the common good.

When we discuss things in math class, it is not a competition. It is a collaborative effort in which we all play a part. The goal of our collaborative effort is to all come to a better understanding of the argument or question we’re exploring. Wherever you are in your understanding of the question, you have a role to play! Asking good questions is just as important as offering good answers to those questions. Students can contribute to a discussion of a mathematical text in lots of ways:

1. Ask a clarifying question – pick some language in the text that you don’t understand and ask what it means.

2. Halt the proceedings – when you don’t know what’s going on, make the group explain it to your satisfaction. This should happen frequently!

3. Challenge a bit of argument – mathematicians are fond of leaving out steps that they think the reader will uncritically accept. Often we need not accept these jumps. When you think an argument doesn’t make sense, ask about it.

4. Ask about the definition of a term – mathematics uses lots of unfamiliar vocabulary. Talking about what words mean is always valuable.
5. Request notation clarification – sometimes the board becomes an ungainly mess. Asking for a pause to consolidate our notation is often a vital precursor to seeing new things.
6. Call something out as important – focusing our discussion on important parts of an argument is always valuable. Even if you're not sure *why* something is important, calling it out as important is worthwhile.
7. Answer a question – there will be lots of opportunities for this. Don't feel compelled to know everything about the answer when you start. Part of an answer moves the discussion forwards.
8. Work a particular case – sometimes it's nice to have particular examples of some general theory we're bandying about. Working out a particular is always valuable.
9. Offer up an analogy – it's often easier to think about mathematical things by analogy. If you think part of the argument reminds you of something else, offer that up to the group.

Similarly, there are lots of ways students (and teachers!) can sabotage a good mathematical discussion:

1. Don't be silent – actively participating in the discussion helps you process the ideas we're examining.
2. Don't dominate – if other people aren't actively participating in the discussion, you're missing out on valuable contributions and you're preventing your brothers and sisters in Christ from processing and understanding what we're talking about. If you *get it*, your goal is to help others walk the same path you walked to get there. Don't spoil it for them! Teachers: this goes doubly for you.
3. Don't be a jerk – we are all brothers and sisters in Christ. We are called to love one another and treat one another as we would be treated. Whatsoever you do to the least of your classmates, that you do unto Christ. A note: being a jerk can be active or passive. Not listening when someone is talking is as selfish as calling them a "poo-head". So be considerate and thoughtful.

Teachers have their own special ways of sabotaging a discussion:

1. If students believe, even for a minute, you will do the thinking for them, they will not think. If you insist that they think, they will call your bluff by sitting in silence. I've never had a group of students make it past about twenty minutes without someone breaking and saying something interesting. Especially early, you must be very patient with students. Even once they've realized that you *do* want them doing the thinking, they will test you on this every so often. Don't give in. For the group that made it to 20 minutes, I reminded them that their paper was due in a week, and it's much easier if they talk about it *together* before they write it. That encouraged them.
2. Seriously though - you have to not help them. It's really hard. They are going to be expert weaselers.
3. If students believe that you only care about the ideas of the best mathematician in the room, then they will wait for that mathematician to do all the work. You'll probably have

a student who just *gets* it. It's important that that student not dominate *your* attention. That means actively engaging with quiet and struggling students, calling out good idea that flow from other students, and sometimes even imposing restrictions on your best mathematician. It's amazing how well they learn to articulate their ideas when they're not allowed to write anything on the board and only allowed to ask questions.

4. If students believe their peers will think less of them for venturing ideas, they will not venture ideas. I mercilessly quash any talk that cuts down a student for expressing wrong ideas. Getting kids to separate their ego from their mathematical ideas takes constant and thoughtful encouragement and can be destroyed in a single ripple of laughter. It's hard to hit the balance where students feel comfortable arguing against genuinely bad ideas without imputing moral blame to the student who has articulated the bad idea. Finding the right balance takes lots of open dialogue and trust. The students must come to know and trust one another as mathematicians, so the classroom must be rooted in Christian love.

Assessing Mathematical Writing

What do mathematicians make? According to G.H. Hardy, "[a] mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."[1] I find that compelling. How do mathematicians communicate their art? They write to one another. Much of mathematics has been transacted over letters and papers. We should join the rest of our sibling humanities and do the same. Once the class has settled on a path through a problem, it's time for each student to write his or her own paper synthesizing and explaining that path as elegantly and clearly as possible.

This terrifies the students the first time it happens – math has usually been the respite from paper writing – so we do the first paper together in class, writing it sentence by sentence. We talk about each sentence, whether it can be whittled down just a little bit more, whether it conveys precisely what we want it to convey. We gradually weave them together into a cogent explanation that would be convincing to someone approaching the problem for the first time.

I give the students a pattern and a couple of model papers. Each paper we write is structurally similar: a statement of the question, the givens, an argument, figures, and an explanation of how to check the results. Mostly they're between two and four pages long. It's surprisingly hard to convince students that math papers are both similar to and different from a literature paper. They are similar – both papers will have a thesis and arguments supporting the thesis. But they're also different. I tell the students that you get one joke per math paper, plus one more for each degree you have above a bachelor's. Some of them laugh. I tell my students that mathematical rhetoric is clean and spare, almost laconic.

For every paper after the first, I insist on a rough draft which I mark up and return. Because the papers are short and because we've worked through the problem in a group, the drafts are usually pretty good. Like a literature teacher, I have a list of common errors that I try to wring out of the students. I put question marks where their arguments become incoherent. I grade for style, spelling, and content. Papers don't have to be perfect to get an A, but they have to be clear and well reasoned.

[1] G.H. Hardy, *A Mathematican's Apology*, p. 13

We get to talk about audience. Seventh graders would need to explain how to solve $3x + 1 = 7$. Tenth graders can gloss over that without losing their audience. We get to talk about clarity and language. If two lengths are different, can you use the same letter to represent them? We get to talk about the role of equations and figures. Is a figure enough without any explanatory text? (Sometimes!)

On to the Problems!

So there's the nuts and bolts. I hope you'll enjoy working through these problems with your students. Ask questions. Play with the ideas. Make mistakes. Try some concrete examples. See what you can come up with!

In book seven of the Republic, Plato notes that:

> for [practical] purpose[s] a very little of either geometry or calculation will be enough; the question relates rather to the greater and more advanced part of geometry – whether that tends in any degree to make more easy the vision of the idea of good; and thither, as I was saying, all things tend which compel the soul to turn her gaze towards that place, where is the full perfection of being, which she ought, by all means, to behold.

If we teach mathematics like one of her peers, the humanities, we can incline our students, ever so gently, towards the vision of the good.

Chapter 1

Playing With Prime Factors

1.1. Student Assignment: Prime Factors

The Problem: How many different ways are there to write 148 as a product of prime numbers? What about 256? 1031? 36,449?

You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

You can easily look up the answer on the internet. That said, if you somehow look up the answer, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 2

Relative Primality and Euler's Totient Function

2.1. Student Assignment: Relative Primality

The Problem: How many numbers are relatively prime to 36,499?

You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

It's not totally obvious to me how you'd use the internet to help you with this. That said, if you somehow look up the answer, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 3

Generalizing The Product of Binomials

3.1. Student Assignment: Generalizing the Product of Binomials

The Problem: We've learned that $(x+1)^2 = x^2 + 2x + 1$. Can we generalize that to work for any positive integer power? What is the value of $(x+1)^3$? What about $(x+1)^4$? What about $(x+1)^n$? Is there some pattern at work here? If so, what is it?

You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

The answer to this question is available via the internet, though you might not know what to call it. Nevertheless, if you try to look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through and see the pattern. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 4

Measuring a Geometric Figure

4.1. Student Assignment: Measuring A Geometric Figure

The Problem: What is the length of the diagonal of a rectangular prism whose sides measure a, b, and c (i.e. the distance from the bottom, left, forward corner to the top, right, back corner)?

You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

The answer to this question is surely available via the internet. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

4.2. Teacher Notes: Measuring A Geometric Figure

The Kairos: This problem requires only the Pythagorean Theorem (Euclid I.47) as preparatory material. I have used it with students in Algebra II and Pre-Calculus. It makes a nice first problem for a Pre-Calculus class. I haven't ever used it as a first problem in Algebra II, so I'm not sure how it would fare there.

Timing and Planning: My fastest Pre-Calculus classes have worked through this in one forty-five minute class period. The median class takes a hair under two forty-five minute periods to work out the answer.

If students have never written a math paper before, I budget one forty-five minute period for in-class writing. Otherwise, I give the paper as homework due two days after the completion of the problem.

An Answer: The answer is, happily, $\sqrt{a^2+b^2+c^2}$, which is a generalization of the Pythagorean theorem into the third dimension. The easiest path through the problem is to see that the hypotenusal length of one face of the prism is a short side of the triangle containing the diagonal of the whole prism, and go from there.

Conversation Notes:

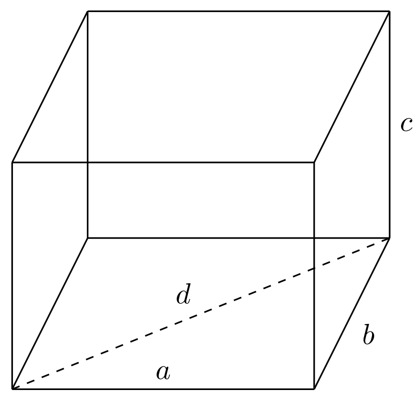
1. Students will wonder whether it is acceptable to have a, b, and c appear in their answer. This breaks two ways: some students will want there to be a single numeric answer. They will be disappointed. Others wonder whether, say h, the hypotenuse of one face, can appear in their answer. The rule I articulate is that their answer must contain only elements present in the original question.
2. Students might attempt to solve the problem in some particular case by getting a box and actually measuring a diagonal with a ruler or string or something. This is an appropriate time to digress for five or ten minutes and converse with them about approximate and exact answers in mathematics. There is a role for approximate work, but it does not have the final say.
3. Students might wonder whether the answer depends on the rotation of the box, i.e. if a and b are swapped, does the answer change. They can work this out on their own, and should. That the answer is symmetric encourages us to believe it.
4. This problem allows for conversation about checking your answer. What happens geometrically and analytically if $c = 0$? Does that support our answer? How so?
5. There's a lovely pattern at work here: in a one dimensional space, the distance would be $\sqrt{a^2}$, which is just $|a|$, what we would expect. Whether this pattern continues in higher dimensions as well is good fodder for a conversation about metric spaces later in the year.

Further Reading and Bibliography: This problem is taken from George Polya's *How to Solve It*, beginning on page 7. The notes in Polya's explanation of how to teach this problem are formative and essential.

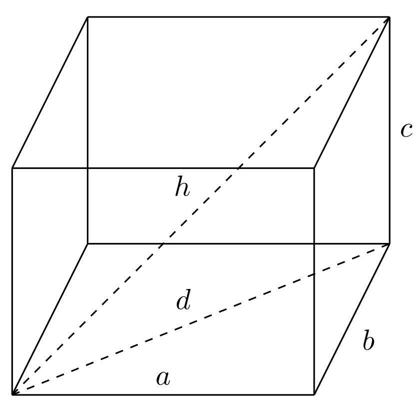
4.3. Sample Writing: Measuring a Geometric Figure

What is the length of the diagonal of a rectangular prism whose sides measure a, b, and c? The length is $\sqrt{a^2 + b^2 + c^2}$.

Consider the base of the prism, the rectangle whose sides are a and b. By the Pythagorean theorem, the diagonal of that rectangle is $\sqrt{a^2 + b^2}$. Let that diagonal be called d.



That diagonal d is also one side of a right triangle that "stands up" in the middle of the rectangular prism. The other short side of that triangle is c. The hypotenuse of that triangle is the length we are trying to find. Happily, we can use the Pythagorean theorem to find that hypotenuse. The short sides of the right triangle are c and d, so our hypotenuse $h = \sqrt{c^2 + d^2}$. But $d^2 = a^2 + b^2$, so the length $h = \sqrt{a^2 + b^2 + c^2}$.



Because a, b, and c are all identical in the formula, our answer doesn't depend on how we rotate the prism, which is good. Also, if we let the height of the prism be zero, so it turns into a rectangle, our formula produces the Pythagorean theorem, also good confirmation that our answer is correct.

Chapter 5

Perpendicular Lines

5.1. Student Assignment: Perpendicular Lines

The Problem: Two lines are perpendicular if their slopes are opposite reciprocals of each other (i.e. the slope of one line is m and the slope of the other is $-\frac{1}{m}$). I remember learning this in maybe sixth or seventh grade, but I never learned **why**. Can we convince ourselves of the truth of this theorem?

You must turn in your explanation on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

The answer to this question is surely available via the internet. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 6

Deriving the Quadratic Formula

6.1. Student Assignment: The Vertex Form of the Parabola

The Problem: We've learned how to find the x-intercepts of a parabola when it's written in vertex form. Fair enough. Sometimes, if we get lucky, we can factor a parabola in standard form and find its x-intercepts. But sometimes factoring it seems hard! Is there any way for us to combine the vertex form and standard form to do something interesting? A good place to start is to take the general vertex form and turn it into standard form.

This is a much more nebulous question than we've played with before, so there won't be a paper on this one, but when you find out the cool thing, it'll eventually turn into a quiz of some sort.

It's not super obvious to me that the internet can help you here. If you try to look up what we're doing, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. And it's a cool pattern that I think you'll find surprising. Also, looking it up on the internet is academic fraud, so don't do that.

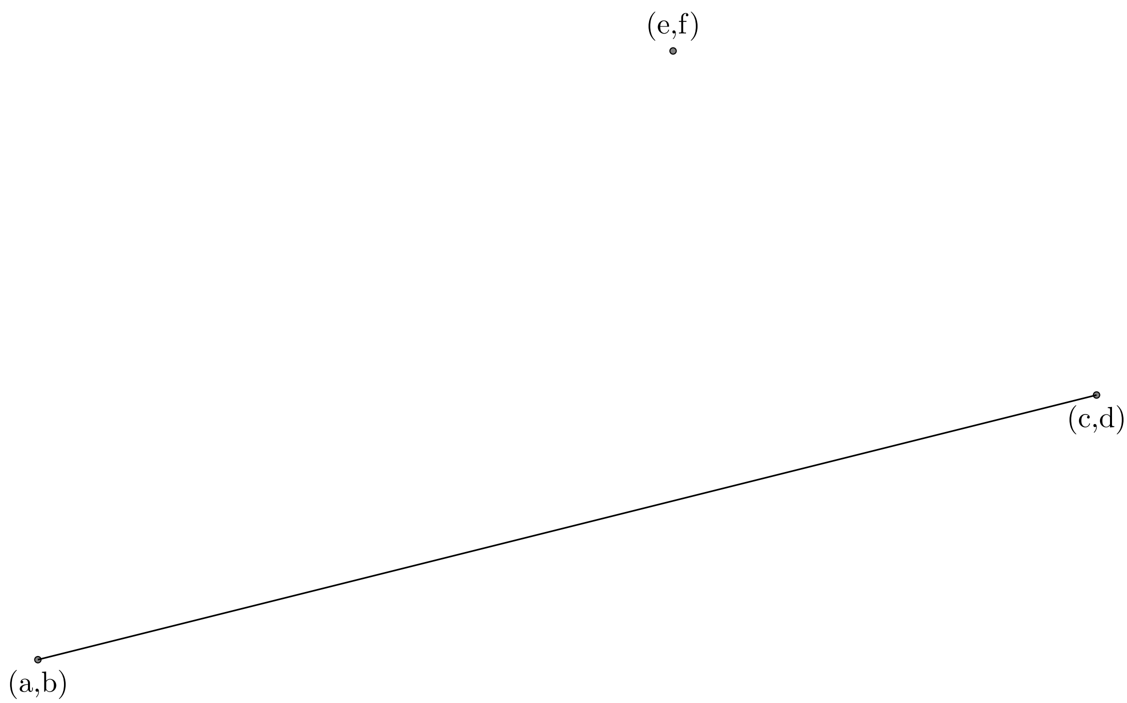
You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**.

Chapter 7

Orthogonal Distance

7.1. Student Assignment: Orthogonal Distance

The Problem: In the following figure, What is the shortest distance from the point to the line segment?



There are at least three entirely different ways of attacking this. At least one answer to this question is easily available via the internet if you know what to look for. You could find it by accident if you're not careful. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

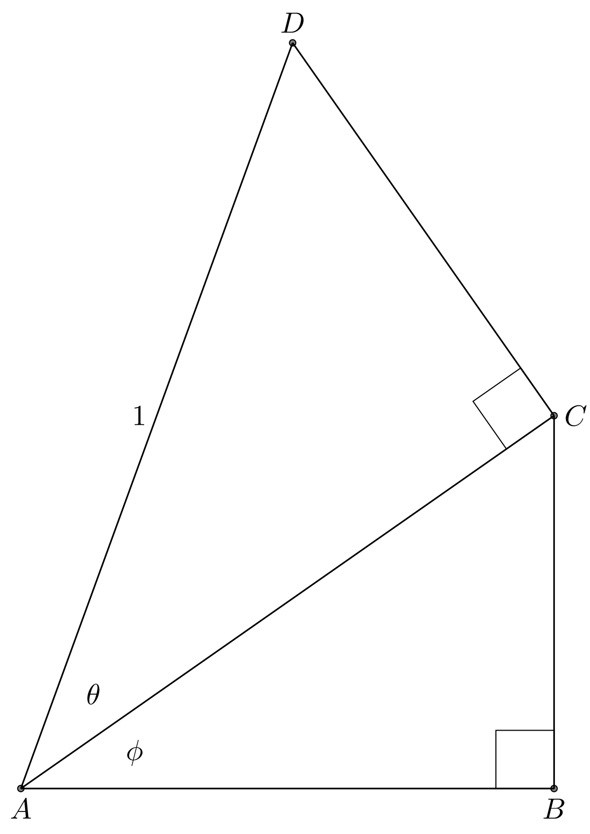
You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 8

Deriving the Angle Sum Formulae

8.1. Student Assignment: A Pair of Stacked Triangles

The Problem: In the following figure, what is the length of the vertical connecting point D to segment AB? There are two ways to express that length, and finding both of them reveals something interesting. Similarly, call the point where that vertical intersects AB point E. What is the length of segment AE? There are two ways to express that length, and finding both of them reveals something interesting! So find both!



The answer to this question is available via the internet. You could find it by accident if you're not careful. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

8.2. Teacher Notes: A Pair of Stacked Triangles

The Kairos: This problem has a narrow window. Students should be comfortable with solving right triangles using trigonometric functions, but should not yet have encountered the angle addition formulae. Working out this problem derives the angle addition formulae, but you shouldn't tell them that!

Timing and Planning: This one usually takes about two forty-five minute periods for a Pre-Calculus class. If they haven't drawn some auxiliary triangles near the middle of the second class, I might nudge them gently in that direction. This doesn't naturally lend itself to a paper (the argument is almost entirely visual), so I usually assess it by making them reproduce the figure and label it on a quiz.

An Answer: Most students quickly leap to the idea that $\overline{DE}$ is $\sin(\theta+\phi)$ and $\overline{AE}$ is $\cos(\theta+\phi)$. Fair enough. Once they say that, ask them whether they can express those lengths without any arithmetic inside the argument to the trigonometric functions. This is trickier, and you might have to give them some examples of what's allowed and what's not allowed. The hope is that they work out that $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

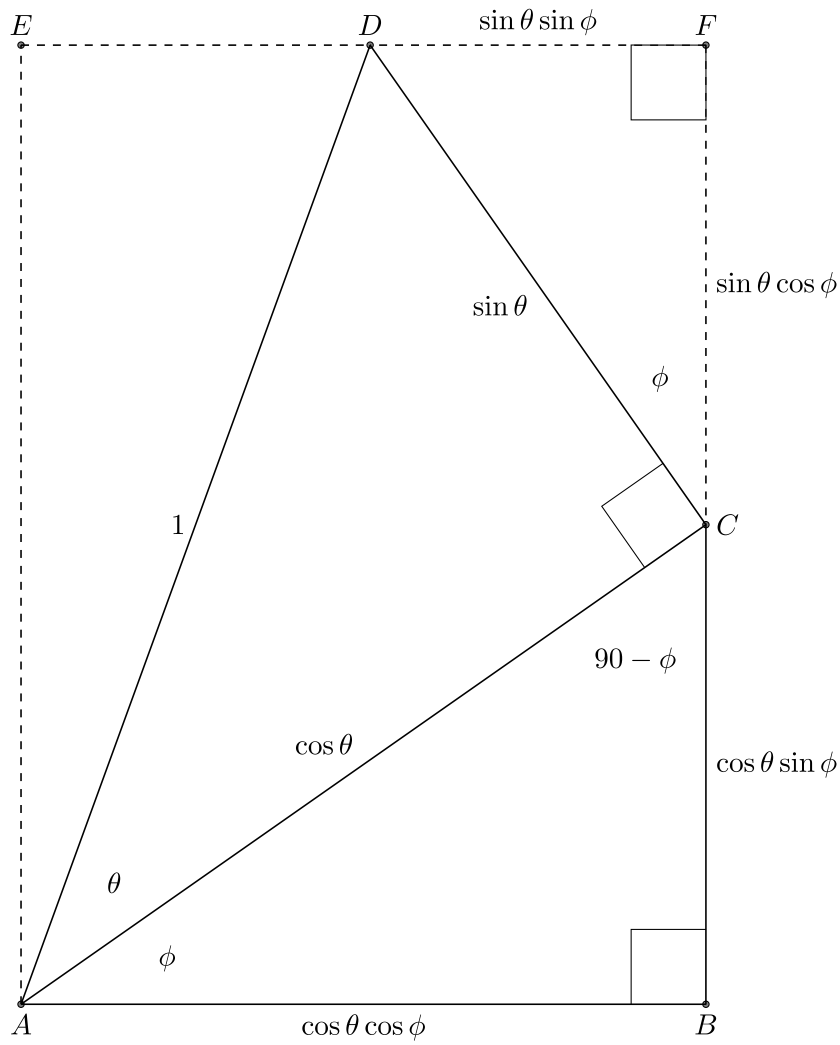
Conversation Notes:

1. This one goes fast, then slow, then fast. Students work out the $\sin(\theta + \phi)$ answer quickly, and are frustrated when I want more than that. But I promise that the idea is neat and worthwhile. They then enter a slow phase of flailing until someone decides to draw the complete rectangle (i.e. add two more right triangles whose hypotenusae are DC and AD). Once those are drawn, it goes fast.
2. This problem punishes classes who don't draw clear and unambiguous diagrams on the board. It's very easy for the diagram to become unwieldy.
3. Students often don't notice that the length of AD is 1. Once or twice I've had to really emphatically encourage them to make sure they are using *all* the givens. Usually they pick up on that on their own.
4. If students get stuck for more than about 15 minutes on day one, which happens, I encourage them to draw more triangles. This is always good advice. I also encourage them to label as many angles as they can. This is also always good advice.
5. Conversely, if you have good geometry students, be prepared for a gaggle of bisected angles and dozens of auxiliary triangles, none of which are helpful at all. I allow this to proceed until it collapses under its own weight. Often students are convinced they need to find where DE intersects AC, which is not at all helpful. I let them play around with it.

Further Reading and Bibliography: The formula dates to Ptolemy, but the presentation we work through is, again, pulled from the Wikipedia's list of trigonometric formulae. There are other proofs of the identity available on the internet, but I find this the clearest.

8.3. Sample Writing: A Pair of Stacked Triangles

I don't assign a paper to accompany this problem. I assess it with a one question quiz (show that $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$) requiring the students to sketch and label the diagram. I've found that that's enough reasoning. The following diagram is what I hope they will produce:

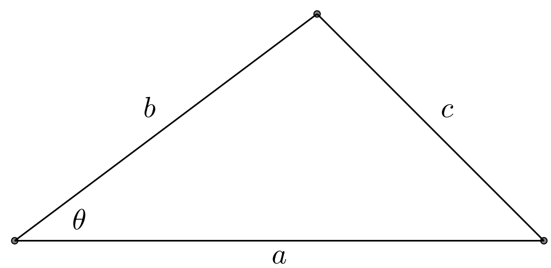


Chapter 9

A Not-Quite Right Triangle

9.1. Student Assignment: A Not-Quite Right Triangle

The Problem: In the following figure, what is the length of c in terms of a, b, and θ? Note that this problem is, like, really easy if $\theta = 90°$. Don't assume that *any* of the angles in the figure are right. I'm interested in the more general case where θ can be *any* angle.



You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

The answer to this question is available via the internet. You could find it by accident if you're not careful. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 10

An Open Problem in Trigonometry?

10.1. Student Assignment: An Open Problem?

In a previous problem we worked through a geometric interpretation of the angle addition formula: $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$. The power reducing formula, $\sin^2 u = \frac{1-\cos(2u)}{2}$, has a well known algebraic proof which you all can work out pretty quickly. Does it also have a nice geometric interpretation?

This may be an open problem. I've never found an answer anywhere I've looked. We'll spend a few classes on this and see whether anything interesting happens.

You must turn in your work on separate paper. Because I don't necessarily expect you to arrive at a solution, your work will be graded on being a good faith effort at the problem.

The answer to this question does not appear to be available via the internet, which is pretty cool, right?

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper. Please don't talk to last year's class about this.

Chapter 11

Adding Up Cubes

11.1. Student Assignment: Adding up Cubes

The Problem: Following in the footsteps of Gauss, we discovered that there is a pattern to the sum of the first n consecutive integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Is there an analogous pattern for the sum of the first n cubic numbers? For example, $1^3 + 2^3 + 3^3 = 36$. What about $1^3 + 2^3 + 3^3 + ... + n^3$?

You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

The answer to this question is available via the internet. You could find it by accident if you're not careful. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

Chapter 12

An Inscribed Pyramid

12.1. Student Assignment: An Inscribed Pyramid

The Problem: A pyramid with a square base is inscribed in a cube of volume 1 such that the tip of the pyramid falls exactly in the middle of the top face of the cube and the base of the pyramid is the bottom face of the cube. What is the volume of the pyramid? For this problem, a good approximation is fine. You know how weird it is for me to be fine with approximation, so pay attention to that!

You might have learned this as a formula at some point in school; I want you to explain why that formula is true!

You must turn in your work on separate paper. It will be graded on clarity of thought, persuasiveness of argument, and neatness of expression. If your paper is manifestly lacking in **any** of those attributes, I will return it to you to be re-worked.

An answer to this question is available via the internet. If you look it up, you waste your time and mine. The value in this question is developing the muscles of thought required to work it through. Also, looking it up on the internet is academic fraud, so don't do that.

You may think about and work on this with other people in the class **during class**. Please do not consult with adults (other than me) or students who have already taken this class. Again, the goal is for you to **figure this out**. If you work with others during class time, you are nevertheless responsible for turning in your own paper.

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