A Companion to Diophantus's Arithmetic

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Using This Text

This is a short book about cultivating virtues. It aims at encouraging curiosity, clarity, and community. Those three, in concert, allow for and encourage mathematical excellence. The way to encourage those virtues in your students is to confuse them: to give them hard problems –problems they don't know how to solve– that will tax their intellectual and creative muscles. This is a selection of problems from Diophantus's Arithemtic, annotated and expanded for use with younger students.

On to the Problems!

So there's the nuts and bolts. I hope you'll enjoy working through these problems with your students. Ask questions. Play with the ideas. Make mistakes. Try some concrete examples. See what you can come up with!

In book seven of the Republic, Plato notes that:

> for [practical] purpose[s] a very little of either geometry or calculation will be enough; the question relates rather to the greater and more advanced part of geometry – whether that tends in any degree to make more easy the vision of the idea of good; and thither, as I was saying, all things tend which compel the soul to turn her gaze towards that place, where is the full perfection of being, which she ought, by all means, to behold.

If we teach mathematics like one of her peers, the humanities, we can incline our students, ever so gently, towards the vision of the good.

To separate a given number into two numbers having a given difference.

Diophantus's Argument

Given number: 100, given difference 40.

[I] Let the lesser number required be x.

[II] The greater number is then $x + 40$.

[III] Then $2x + 40 = 100$.

[IV] Therefore $x = 30$.

Thus the required numbers are 30 and 70.

Questions to Ponder

How many givens does this postulate require?

When Diophantus represents the greater number as $x + 40$, which of the givens is he making use of?

Diophantus does not explain how he moves from step three to step four? Can you explain that?

Could you fulfill the requirements if you made the *greater* number x? How would the argument change?

Practice With the Method

To separate a given number into two numbers having a given difference:

Given number: 80, given difference 20. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

Given number: 70, given difference 15. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

Given number: 50, given difference 12. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

Given number: 19, given difference $2\frac{3}{4}$. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

1. Given number: 10, given difference: 4. Lesser number: _______ Greater number: _______
2. Given number: 20, given difference: 8. Lesser number: _______ Greater number: _______
3. Given number: 9, given difference: 3. Lesser number: _______ Greater number: _______
4. Given number: 11, given difference: 3. Lesser number: _______ Greater number: _______
5. Given number: 13, given difference: 3. Lesser number: _______ Greater number: _______
6. Given number: 13, given difference: 5. Lesser number: _______ Greater number: _______
7. Given number: 15, given difference: 5. Lesser number: _______ Greater number: _______

8. Given number: 9, given difference: 2. Lesser number: ______ Greater number: ______

9. Given number: 21, given difference: 6. Lesser number: ______ Greater number: ______

10. Given number: 20, given difference: $\frac{1}{2}$. Lesser number: ______ Greater number: ______

11. Given number: $\frac{7}{8}$, given difference: $\frac{1}{5}$. Lesser number: ______ Greater number: ______

12. Given number: $4\frac{3}{8}$, given difference: $1\frac{2}{3}$. Lesser number: ______ Greater number: ______

0.1. Does it Generalize?

When we tackle problems like this, we're often interested in whether we can solve *all* problems that have the same shape. In other words, can we generalize this problem? Let's start by generalizing over the difference:

To separate a given number into two numbers having an unknown given difference:

Given number: 10, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

Given number: 20, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

Given number: 100, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

We could generalize even further! Let's make *both* the given number and the given difference general.

Given number: **s**, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

To separate a given number into two having a given ratio.

Diophantus's Argument

Given number: 60, given ratio 3:1.

[I] Let there be two numbers x, $3x$.

[II] Therefore $x = 15$.

Thus the required numbers are 45 and 15.

0.2. Questions to Ponder

This argument is actually a little simpler than his first. What's the only difference int he wording? How does that translate into symbols?

Diophantus again leaves out some reasoning. What happens between steps [I] and [II]?

Could you fulfill the requirements if you made the *greater* number x? How would the argument change?

Practice With the Method

To separate a given number into two numbers having a given ratio:

Given number: 30, given ratio 2:1. Let there be two numbers ____ , ____ . Therefore ____ = ______ . Thus the required numbers are ____ and ____ .

Given number: 50, given ratio 4:1. Let there be two numbers ____ , ____ . Therefore ____ = ______ . Thus the required numbers are ____ and ____ .

Given number: 50, given ratio 5:1. Let there be two numbers ____ , ____ . Therefore ____ = ______ . Thus the required numbers are ____ and ____ .

Given number: 8, given ratio 5:1. Let there be two numbers ____ , ____ . Therefore ____ = ______ . Thus the required numbers are ____ and ____ .

1. Given number: 10, given ratio: 4. Lesser number: ______ Greater number: ______
2. Given number: 20, given ratio: 8. Lesser number: ______ Greater number: ______
3. Given number: 9, given ratio: 3:1. Lesser number: ______ Greater number: ______
4. Given number: 9, given ratio: 1:3. Lesser number: ______ Greater number: ______
5. Given number: 13, given ratio: 5:1. Lesser number: ______ Greater number: ______
6. Given number: 13, given ratio: 1:5. Lesser number: ______ Greater number: ______
7. Given number: 15, given ratio: 5:1. Lesser number: ______ Greater number: ______
8. Given number: 9, given ratio: 2:1. Lesser number: ______ Greater number: ______
9. Given number: 21, given ratio: 6:1. Lesser number: ______ Greater number: ______
10. Given number: 20, given ratio: 1:2. Lesser number: ______ Greater number: _____

11. Given number: $\frac{7}{8}$, given ratio: 1:5. Lesser number: _______ Greater number: ______

12. Given number: $4\frac{3}{8}$, given ratio: 1:2. Lesser number: ______ Greater number: ______

Does it Generalize?

When we tackle problems like this, we're often interested in whether we can solve *all* problems that have the same shape. In other words, can we generalize this problem? Let's start by generalizing over the difference:

To separate a given number into two having a given ratio.

Given number: 50, given ratio **r**:1. Let there be two numbers _____ , _____ . Therefore _____ = _______ . Thus the required numbers are _____ and _____ .

Given number: 20, given ratio **r**:1. Let there be two numbers _____ , _____ . Therefore _____ = _______ . Thus the required numbers are _____ and _____ .

Given number: **n**, given ratio 3:1. Let there be two numbers _____ , _____ . Therefore _____ = _______ . Thus the required numbers are _____ and _____ .

Given number: **n**, given ratio 5:1. Let there be two numbers _____ , _____ . Therefore _____ = _______ . Thus the required numbers are _____ and _____ .

We could generalize even further! Let's make *both* the given number and the given difference general.

Given number: **n**, given ratio **r**:1. Let there be two numbers _____ , _____ . Therefore _____ = _______ . Thus the required numbers are _____ and _____ .

To divide a given number into two numbers such that one is a given ratio of the other plus *a given difference.*

Diophantus's Argument

Given number: 80, given ratio 3:1, difference 4.

[I] Let the lesser number be x.

[II] Therefore the larger is 3x + 4, and

[III] 4x+4 = 80, so that x = 19

Thus the required numbers 61 and 19.

0.3. Questions to Ponder

Diophantus again leaves out some reasoning. What happens between steps [II] and [III]?

If the given difference is odd, will the lesser and greater numbers be whole numbers? Why or why not?

Could you fulfill the requirements if you made the *greater* number x? How would the argument change?

Practice With the Method

To divide a given number into two numbers such that one is a given ratio of the other *plus* a given difference.

Given number: 30, given ratio 2:1, given difference 4. Let the lesser number be _____ . Therefore the larger is ______________ and ___________________ = _________ , so that _____ = _____ . Thus the required numbers are _____ and _____ .

Given number: 50, given ratio 2:1, given difference 4. Let the lesser number be _____ . Therefore the larger is ______________ and ___________________ = _________ , so that _____ = _____ . Thus the required numbers are _____ and _____ .

Given number: 50, given ratio 2:1, given difference 8. Let the lesser number be _____ . Therefore the larger is ______________ and ___________________ = _________ , so that _____ = _____ . Thus the required numbers are _____ and _____ .

Given number: 80, given ratio 3:1, given difference 5. Let the lesser number be _____ . Therefore the larger is ______________ and ___________________ = _________ , so that _____ = _____ . Thus the required numbers are _____ and _____ .

1. Given number: 10, given ratio: 2, given difference 2.
   Lesser number: _______ Greater number: _______

2. Given number: 10, given ratio: 2, given difference 3.
   Lesser number: _______ Greater number: _______

3. Given number: 12, given ratio: 2, given difference 4.
   Lesser number: _______ Greater number: _______

4. Given number: 16, given ratio: 2, given difference 4.
   Lesser number: _______ Greater number: _______

5. Given number: 20, given ratio: 2, given difference 4.
   Lesser number: _______ Greater number: _______

Does it Generalize?

When we tackle problems like this, we're often interested in whether we can solve *all* problems that have the same shape. In other words, can we generalize this problem? Let's start by generalizing over the difference:

To divide a given number into two numbers such that one is a given ratio of the other *plus* a given difference.

Given number: 80, given ratio 3:1, given difference **d**. Let the lesser number be ____ . Therefore the larger is ______________ and __________________ = _________ , so that ____ = ____ . Thus the required numbers are ____ and ____ .

Given number: 80, given ratio **r**:1, given difference 4. Let the lesser number be ____ . Therefore the larger is ______________ and __________________ = _________ , so that ____ = ____ . Thus the required numbers are ____ and ____ .

We could generalize even further! Let's make *both* the given ratio and given difference general:

Given number: 80, given ratio **r**:1, given difference **d**. Let the lesser number be ____ . Therefore the larger is ______________ and __________________ = _________ , so that ____ = ____ . Thus the required numbers are ____ and ____ .

To find two numbers in a given ratio such that their difference is also given.

Diophantus's Argument

Given ratio: 5:1, given difference 20.

[I] Let the lesser number be x, the greater 5x.

[II] Therefore 4x = 20, x = 5.

Thus the required numbers are 25,5.

Questions to Ponder

Though this is a simpler argument than the previous, Diophantus again leaves out some reasoning. What happens between steps [I] and [II]?

Could you fulfill the requirements if you made the *greater* number x? How would the argument change?

0.4. Practice With the Method

To find two numbers in a given ratio such that their difference is also given.

Given ratio: 5:1, given difference 20. Let the numbers be x, ____ .

Therefore ____ = ____ , x = ____ . Thus the required numbers ____ , ____ .

Given ratio: 4:1, given difference 15. Let the numbers be x, ____ .

Therefore ____ = ____ , x = ____ . Thus the required numbers ____ , ____ .

Given ratio: 3:1, given difference 18. Let the numbers be x, ____ .

Therefore ____ = ____ , x = ____ . Thus the required numbers ____ , ____ .

1. Given ratio: 4, given difference 2.
   Lesser number: _______ Greater number: _______

2. Given ratio: 12, given difference 2.
   Lesser number: _______ Greater number: _______

3. Given ratio: 2, given difference 12.
   Lesser number: _______ Greater number: _______

4. Given ratio: 5, given difference 7.
   Lesser number: _______ Greater number: _______

5. Given ratio: 8, given difference 3.
   Lesser number: _______ Greater number: _______

Does it Generalize?

When we tackle problems like this, we're often interested in whether we can solve *all* problems that have the same shape. In other words, can we generalize this problem? Let's start by generalizing over the difference:

To divide a given number into two numbers such that one is a given ratio of the other *plus* a given difference.

Given ratio: 4:1, given difference **d**. Let the numbers be x, _____ .

Therefore _____ = _____ , x = _____ . Thus the required numbers _____ , _____ .

Given ratio: **r**:1, given difference **4**. Let the numbers be x, _____ .

Therefore _____ = _____ , x = _____ . Thus the required numbers _____ , _____ .

We could generalize even further! Let's make *both* the given ratio and given difference general:

Given ratio: **r**:1, given difference **d**. Let the numbers be x, _____ .

Therefore _____ = _____ , x = _____ . Thus the required numbers _____ , _____ .